Implied Volatility for FX Vanilla Options : Back to Basic Principles

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Summary

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Introduction

We revisite the pricing of plain vanilla FX option using liquid instruments for market coherent estimation. This paper discusses the volatility surface construction from option trading strategies which provides a way to interpolate that volatility at any strike from the surface. We show how to derive a strike and its corresponding Implied Volatility out of the core region and uses the well-known non free arbitrage criterias to their consistency. Finally, we see how to price the plain vanilla FX option with any strike between the extra lower and upper points.

This note is structured as follow: in section 1, we point out a general concept of deal with FX currencies while in section 2, we provide the plain vanilla FX option pricing methodology. Finally, sections 3 and 4 discuss about the Implied Volatility interpolation at any strikes and maturities.

1 FX Rate Convention

We briefly introduce in that section the FX terminology before explain the FX option vanilla pricing within the next section. Currencies pairs, under ISO codes, are quoted in the format FOR-DOM, where FOR and DOM denoted foreign and domestic currency respectively. As underline in the [1], *the notion of foreign and domestic does not refer to the location of the trading entity, but only to this quotation convention.* In order words, under the FX spot rate X_t =FOR-DOM, one understands that one unit of foreign currency corresponds to FOR-DOM units of the domestic currency. For example, in EURUSD, the EUR denotes foreign currency while USD denotes the domestic currency. Thus, the rate of EURUSD indicates the value of one Euro in USD. For that purpose, we will refer to the "domestic" currency as of a base or numeraire currency in relation to which "foreign" amount of money is measured. For a more detailed about the FX terminology, we refer the reader to [2, 3, 4]

2 FX European Vanilla Option Pricing

We denoted $\sigma(K,T)$ the Implied Volatility for which the Black-Scholes formula matches the market price $\Pi(K,T)$ for an european vanilla option under a nominal of one domestic currency unit in foreign currency, a strike of K and expiry date T.

Under the domestic risk neutral (as a reminder, a domestic currency is a numeraire) assumption and that the underlying currency pair follows the lognormal dynamic:

$$\frac{dX_t}{X_t} = \left(r_d - r_f\right) dt + \sigma dW_t^d \tag{1}$$

where r_d and r_f is respectively domestic and foreign risk free rates and are supposed to be deterministic interest rate, σ is the deterministic volatility, W_t^d is a standard brownian motion and X_t the currency pair rate.

An european option premium $\Pi(K,T)$, in domestic currency, of the contract at expiry date T is:

$$\begin{aligned}\Pi(K,T) &= e^{-r_d(T-t)}\mathbb{E}_t^{Q^T}\left[\{\zeta(X(T)-K)\}^+\right]\\ &= \zeta\left[e^{-r_f(T-t)}X(T)\Phi(\zeta d_+) - Ke^{-r_d(T-t)}\Phi(\zeta d_-)\right]\\ &= \zeta e^{-r_d(T-t)}\left[F_{t,T}\Phi(\zeta d_+) - K\Phi(\zeta d_-)\right]\end{aligned} \tag{2}$$

where:

$$\begin{aligned}F_{t,T} &= e^{\left(r_d-r_f\right)(T-t)}X(T)\\ d_+ &= \frac{ln\left(\frac{F_{t,T}}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\\ d_- &= d_+ - \sigma\sqrt{T-t}\end{aligned} \tag{3}$$

$F_{t,T}$ is the forward rate value of the FX rate at delivery date T seen at time t while $\Phi(x)$ is the cumulative normal density function at point x, $T-t$ is the time remaining to maturity and $\zeta = +1$ for a Call and -1 for a put. The currency in which the option values are measured is called the **premium** or **accounting** currency. The **notional** is the amount of currency which the holder is entitled to exchange. The above formula is retrieved to one unit of foreign notional, with a value in units of domestic currency.

The option price defined in equation 2 is expressed in domestic currency, also known as the domestic pips price. However, there exist other FX option price quotations with the relationship style summarized in the follwing table:

Designation	Relation style
Option premium in foreign currency	$\Pi_f(K,T) = \frac{\Pi(K,T)}{XK}$
Option premium in percentage foreign	$\Pi_{\%f}(K,T) = \frac{\Pi(K,T)}{X}$
Option premium in percentage domestic	$\Pi_{\%d}(K,T) = \frac{\Pi(K,T)}{K}$

Table 1: Premium market quoting convention

It is straightforward to derive the option price from the equations 2 and relation disclosed above. However, it is worth mentioned that these technics of constructing these differents quote work only if there are two notionals given by strike: $K = \frac{N}{\hat{N}}$ where N is the notional in domestic currency while $\hat{N}$ is the notional in foreign currency. As underline in [5], *these relations are true for European and American style vanilla options, even in the presence of barriers and accrual features, but is most definitely note true for digital options.* We also refer the reader to [6] for more details about the validity of the above relationships.

Unlike the Equity markets, FX markerts quote Delta sensitivity and volality of Straddle, Risk Reversal and Butterfly (from which a practitioner deduces the Implied Volatility) instead of Implied Volatility. For that purpose, the following sub-section will focus on sensitivity-Delta quoting

2.1 Sensitivity-Delta Quoting Convention

This sub-section gives a brief introduction to FX Delta convention. We follow the same brief summary as provided in [5, 7]. In fact, regarding the sensitivity delta, it is important to use a definition consistent with the market convention for the currency. In other words, it is crucial to know how the deltas are definied when such quotes are obtained. Otherwise, option prices will be incorrect. As for FX option vanilla premium market quoting convention, there are several definition of delta such as: spot/forward delta, pips/percentage delta.

2.1.1 Pips Spot Delta or Unadjusted Pips Spot Delta

The pips spot delta is the sensitivity of the present value with respect to the spot rate. This style of Delta implies that the premium currency is in domestic while the notional currency is in foreing. In the market, this style is commonly adapted by currency pairs with USD as domestic currency such as: EURUSD, AUDUSD, etc. The pips spot delta is equivalent to the standard Black-Scholes delta:

$$\Delta = \zeta e^{-r_f(T-t)} \Phi(\zeta d_+) \tag{4}$$

2.1.2 Pips Forward Delta or Unadjusted Pips Forward Delta

The pips forward delta is the sensitivity of the present value with respect to the forward. As of the pips spot delta, this style of delta implies that the premium currency is domestic and notional currency is foreign. The pips forward delta is often used in FX smile because of the fact that the delta of the Call and the absolute value of the delta of the corresponding Put add up to 100%, i.e the 25%-Delta Call must have the same volatility as a 75%-Delta Put.
In the Black-Scholes model, the pips forward delta has the form:

$$\Delta = \zeta \Phi(\zeta d_+) \tag{5}$$

2.1.3 Premium Adjusted Pips Spot Delta or Percentage Spot Delta

The percentage spot delta or premium adjusted pips spot delta is the first derivative of the present value with respect to the spot, both in percentage foreign while risk exposure is in domestic currency. That delta has the form:

$$\Delta(\%) = \zeta e^{-r_f(T-t)} \frac{K}{F_{t,T}} \Phi(\zeta d_-) \tag{6}$$

2.1.4 Premium Adjusted Pips Forward Delta or Percentage Forward Delta

The premium adjusted pips forward delta is the first derivative of the present value with respect to the forward. As in the case of a premium adjusted pips spot delta, a premium in foreign currency leads to an adjustment of the forward delta. That delta has the form:

$$\Delta(f,\%) = \zeta \frac{K}{F_{t,T}} \Phi(\zeta d_-) \tag{7}$$

At the end, one can see that if the premium currency is in domestic currency, then no premium adjustment is applied and the pips delta is used, whereas if the premium currency is foreign currency, the premium adjusted delta takes care of the correction induced by payment of the premium in foreign currency.

2.2 Construction of Implied Volatility Smiles

The previous subsections introduced FX specific delta. This knowledge is crucial to understand the volatiity construction procedure. In FX markets, the convention is that the market quotes volatilies instead of direct prices of the option. For that purpose, Straddle, Risk Reversal (*RR*) and Butterfly (*BB*) are some of the most traded option strategies from which FX Implied Volatility is derived.

2.2.1 Straddle Strategy

A Straddle is a combination of a Call and a Put with the same underlying, strike K and expiry date T such that the delta of the Strategy has zero value. Such product allows to retrieve the At-The-Money (ATM) Implied Volatility of the FX option such that the ATM Implied Volatility is quoted against a conventional defined ATM strike K_{ATM}. Thereby, a market consistent volatility smile $\sigma(K)$ must admit the fact that $\sigma(K_{ATM}) = \sigma_{ATM}$
Under the unadjusted pips spot/forward delta, and as the delta of the Straddle worst to zero value, one get :

$$K_{ATM} = F_{t,T} e^{\frac{1}{2}\sigma^2_{ATM}(T-t)} \tag{8}$$

For currency pairs including emerging currency (Latin americain such as: MXN, BRL, etc.) and/or options with long maturities (typically greater than 10Y), it is a convention to use the following definition of the ATM strike:

$$K_{ATM} = F_{t,T} \tag{9}$$

2.2.2 Risk Reversal

In FX market, the Risk Reversal is quoted as the difference between the volatility of the Call and of the Put with a designated Delta. Typically, there are two levels of out-of-the money Delta quotation: 25% and 10%. The market consistent smile function $\sigma(K)$ has to match the information implied within the Risk Reversal. Consequently, a Risk Reversal with y ($i.e$ $y = \{25\%, 10\%\}$) level has to fulfill:

$$\sigma_{RR}(y) = \sigma\left(K_{y,c}\right) - \sigma\left(K_{y,p}\right) \tag{10}$$

where $\sigma\left(K_{y,c}\right)$, $\sigma\left(K_{y,p}\right)$ and σ_{RR} are respectively the Implied Volatility of Call with y-level of delta, the Implied Volatility of Put with y-level of delta and the Risk Reversal quote with y-level of delta.
The Risk Reversal measures the skewness of the smile and can be seen as the extra volatility to be added to the y-delta Put volatility compared to a Call volatility with the same absolute delta. Finally, Equation 10 tells us that the Risk Reversal market strike is computed in Black-Sholes model with a single constant volatility of σ^y_{ATM}, such that at these strikes, the Call and Put have respectively 25% and -25% delta.

2.2.3 Butterfly Strategy

The Butterfly is quoted as a single volatility σ_{BB} for delta level, typically 10% and 25% delta.
For a delta level, the associated strikes for the Call and the Put are both computed in Black-Sholes model with a single constant volatility. Under a market consistent volatility $\sigma(K)$, the butterfly quote is expressed as:

$$\sigma_{BB}(y) = \frac{1}{2}\left(\sigma_{y,c} + \sigma_{y,p}\right) - \sigma_{ATM} \tag{11}$$

This quotation leads to a lot of confusion among academics and practitionners. We refer the reader to the documents [8, 9, 10]

2.2.4 Implied Volatility

From equation 10 and 11, the market consistent volatilities of the Call and the Put can be write as:

$$\begin{cases} \sigma_{10\%,p} = \sigma_{BB}(10\%) + \sigma_{ATM} - \frac{1}{2}\sigma_{RR}(10\%) \\ \sigma_{25\%,p} = \sigma_{BB}(25\%) + \sigma_{ATM} - \frac{1}{2}\sigma_{RR}(25\%) \\ \sigma_{25\%,c} = \sigma_{BB}(25\%) + \sigma_{ATM} + \frac{1}{2}\sigma_{RR}(25\%) \\ \sigma_{10\%,c} = \sigma_{BB}(10\%) + \sigma_{ATM} + \frac{1}{2}\sigma_{RR}(10\%) \end{cases} \tag{12}$$

Their respective strikes $K_{y,c}$ and $K_{y,p}$ are determined so as to match the related quoted $y-$delta level. For example, under unadjusted pips spot delta quotation, the strike of the Put option can be retrieved by resolving the following equation:

$$y = -e^{-r_f(T-t)}\Phi\left(-\frac{ln\left(\frac{F_{t,T}}{K_{y,p}}\right) + \frac{1}{2}\sigma^2_{y,p}(T-t)}{\sigma_{y,p}\sqrt{T-t}}\right) \tag{13}$$

where y takes values of -25% and -10%.
The resolution of equations 13 requires the inversion of the cumulative normal density function, which is avalaible under Python langage, otherwise, it can be implemented with a dichotomie, bisection or Newton algorithm or other sophistical method.

3 Volatility Interpolation and Extrapolation in the Strike Axis

From the previous section, we retrieved five usual couples of strikes and volatilities. This section aims at defining the Implied Volatility for a random strike level inside a market zone and outside of that zone given liquid expiring date t.

3.1 Implied Volatility interpolation

Various different interpolation methods can be considered, among which SABR model [11], Kernel methods, stochastic and local volatility models. In this work, two main criteria were used in selecting the interpolation methods. The first criterion is simplicity of implementation and the second one is its robustness, so complex methods that required different type of fit depending on the data are excluded. Furthermore, the selected method had to work with only five data which meant that regression models were also ruled out. Having said that, the cubic spline fulfilled both criteria and will be used as third order polynomial to perform a perfect fit of a five volatility of a smile. The parameterization of that polynomial is as follows:

$$\sigma^2_j(K) = z_j(K) = \sum_{i=0}^{3}\epsilon_i\left(K - \ln\frac{F}{K_j}\right)^i \text{ for K in } \left[K_j, K_{j+1}\right] \tag{14}$$

where ϵ_i is the coefficient of the polynomial.
The index j is associated to the following couple of Implied Volatility and strike:

Index j	Strike (K_j)	Implied Volatility (σ_j)
1	$K_{10\%,p}$	$\sigma_{10\%,p}$
2	$K_{25\%,p}$	$\sigma_{25\%,p}$
3	K_{ATM}	σ_{ATM}
4	$K_{25\%,c}$	$\sigma_{25\%,c}$
5	$K_{10\%,c}$	$\sigma_{10\%,c}$

Table 2: The index and couple of strike and Implied Volatility to be used within the strike interpolation

3.2 Implied Volatility Extrapolation

Numerous papers [12, 13, 14, 15, 16, 17, 18] have addressed the conditions that may be sufficient to ensure that implied volatility is free of arbitrage. For that purpose, in [12], the authors have adressed two methods to perform the implied volatility extrapolation. The first method is based on the interpolation within the region of observed prices, while the implied volatility is setted to be a constant outside of market quotation. The second method is based on the parametric approach (ex.: SABR) where a same methodology is used both within the core and outside regions.
In this paper, we use cubic interpolation in the region of observed prices while the SABR model [11] is used for lower points. For that purpose, outside the core region, the extrapolation is done at extra lower points by employing the below analytical formula:

$$\begin{cases} K_0 = 4K_1 - 3K_3 < K_1; z_0(K_0) = \sigma_{Sabr}(K_0) \\ K_6 = 4K_5 - 3K_3 > K_5; z_6(K_6) = \sigma_{Sabr}(K_6) \end{cases} \tag{15}$$

where $\sigma_{Sabr}(K_0)$ is the unique value of the Implied Volatility yields the option's price under SABR model.
It is easy to check that K_0 is lower than K_1, while K_6 is upper than K_5.

4 Temporel Interpolation

A pragmatic choice used to interpolate the Implied Volatility, given a strike is to assume a flat forward volatility in time. This is equivalent to a linear interpolation in total variance.
Assuming $\sigma_{ATM}(p)$ and $\sigma_{ATM}(q)$ at maturity p and q respectively, we may interpolate an ATM volatility at a time t, i.e $p < t < q$ by the following formula:

$$\sigma^2_{ATM}(t)\,t = \frac{q-t}{q-p}\sigma^2_{ATM}(p)\,p + \frac{t-p}{q-p}\sigma^2_{ATM}(q)\,q \qquad (16)$$

The temporal interpolation in ∓ 10 Delta and ∓ 25 Delta volatilities may follow this methodology.

Conclusion

We have rescaled the FX rate convention to avoid misunderstanding on the FX market quotations. Then the paper outlined various delta quotations commonly used in FX market as options can be traded in both currencies. The common market instruments used to calibrate the Implied Volatility smile are introduced and efficient market calibration is used. Thus, the strike-wise interpolation as well as the extrapolation is outlined to compute the Implied Volatility. Finally, the time-wise interploation is highlight to define the Implied Volatility for a random time.
Follow-up paper will extend Black-Scholes model to price the FX quanto option and describe an efficient way to retrieve the correlation currencies pairs.

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